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Three fermionic generations on a topological defect in extra dimensions.

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We suggest a mechanism explaining the origin of three generations of the Standard Model fermions from one generation in a higher-dimensional theory. Four-dimensional fermions appear as zero modes trapped in the core of a topological defect with topological number three. We discuss hierarchical pattern of masses and mixings which arises in these models.

1 Introduction

Recently, the interest has been renewed to particle physics models in more than four spacetime dimensions (see, for instance, Refs.[1]). These models provide an interesting framework for solving hierarchy problems of the Standard Model of particle interactions. In particular, it has been pointed out [3] that by introducing extra dimensions it is possible to explain the hierarchical fermionic mass pattern. The Standard Model fermions as well as Higgs boson are represented by localized modes in extra dimensions [2], and their effective four-dimensional Yukawa couplings are determined by overlaps between Higgs and fermionic wave functions. To produce hierarchical structure in overlaps, hence in four-dimensional Yukawa couplings, it has been proposed to localize Higgs and three fermionic generations at different points in extra dimensions [3].

In this paper, we follow a completely different approach to produce fermionic spectrum via wave function overlaps, explaining simultaneously the origin of three generations of fermions with similar quantum numbers. Suppose one has single fermionic generation in a multi-dimensional theory. Let us consider a topological defect whose core corresponds to our four-dimensional world. Chiral fermionic zero modes may be trapped in the core due to specific interaction with the fields which build up the defect [4, 5, 2]. In some cases, the index theorem guarantees that the number of chiral zero modes is equal to the topological number of the defect (see for instance Refs.[6]). We will use this property to obtain three fermionic generations localized on a defect with topological number three while having only one generation in the bulk. If the Higgs scalar couples to the defect, it can also be trapped in the core [5, 2]. Hierarchy between masses of three fermionic modes arises due to their different profiles in extra dimensions.

To be specific, we will work with the simplest topological defect which can be characterized by an integer topological number, the global vortex (see Sec. 2). We first recall the mechanism of localization of fermions (Sec. 3) and scalars (Sec. 4), then explain the origin of different Yukawa couplings for similar fermions of three generations (Sec. 5). To explain mixing angles between fermions of different generations, we then introduce a complication in the model (Sec. 6). We present a general discussion of the presented class of models in Sec. 7. Notations and technical details

are outlined in Appendices.

2 A global vortex with topological number k.

Consider a theory of a complex scalar field Φ in six dimensions,

$$\mathcal{L}_\Phi = |\partial_A \Phi|^2 - \frac{\lambda}{2}\left(|\Phi|^2 - v^2\right)^2 . \tag{1}$$

(See Appendix A for notations). The global $U(1)_g$ symmetry $\Phi \to \Phi \mathrm{e}^{i\alpha}$ is broken spontaneously by the vacuum expectation value $|\Phi| = v$.

Let us consider field configurations which do not depend on x_μ. We introduce polar coordinates (r,θ) in (x_4, x_5) plane. There are solutions to the classical field equations which have the form

$$\Phi = v\mathrm{e}^{ik\theta} F(r), \quad k = \pm 1, \pm 2, \ldots . \tag{2}$$

Their topological numbers are defined by the winding numbers k; the function $F(r)$ satisfies the following boundary conditions:

$$F(r) \to 1, r \to \infty;$$

$$F(r) \to 0, r \to 0,$$

and the ordinary differential equation,

$$F'' + \frac{1}{r}F' - \frac{k^2}{r^2}F - \lambda v^2 F\left(F^2 - 1\right) = 0$$

(hereafter prime denotes the derivative with respect to r; we assume $k > 0$). Analytical solution to this equation is unknown. However, from this equation, it follows that

$$\begin{gathered} F(r) = O(r^k),\ r \to 0; \\ F(r) = 1 - \tfrac{k^2}{2\lambda v^2 r^2} + O(r^{-4}),\ r \to \infty. \end{gathered} \tag{3}$$

So, the field configuration describes a "3-brane" in six-dimensional spacetime, located at $x_4 = x_5 = 0$, of radial size of order $\left(\sqrt{\lambda} v\right)^{-1}$.

Energy of this configuration diverges logarithmically at large distances, $r \to \infty$. This can be cured either by gauging $U(1)_g$ (the case of local, or Abrikosov–Nielsen–Olesen vortex) or by introducing some cutoff at large distances (for example, putting

an "anti–vortex" with winding number $(-k)$ far away from the vortex). In any case, this would improve the behavior of energy density at large distances. Here, we are interested in physics inside the core which remains intact after these modifications.

3 Fermionic modes.

Consider a six-dimensional fermionic field Ω which has an axial charge 1/2 under $U(1)_g$,

$$\Omega \to \mathrm{e}^{i\frac{\alpha}{2}\Gamma_7}\Omega \ ,$$

and interacts with Φ via axial Yukawa coupling,

$$\mathcal{L}_\Omega = i\bar{\Omega}\Gamma^A\partial_A\Omega - \left(g\Phi\bar{\Omega}\frac{1-\Gamma_7}{2}\Omega + \text{h.c.}\right) \ . \qquad (4)$$

We will assume that g is real: its complex phase can be rotated away by an axial transformation of the fermion.

Let us perform Fourier transform with respect to four-dimensional coordinates x^μ,

$$\Omega(x_\mu; x_4, x_5) = \frac{1}{(2\pi)^{3/2}}\int d^4k \mathrm{e}^{-ik_\mu x^\mu}\Psi(k_\mu; x_4, x_5) \ ,$$

The Dirac equation following from the Lagrangian Eq.(4) takes the form of Schrödinger type equation for Ψ in the vortex background (2),

$$k_0\Psi = C_i k_i \Psi + D\Psi \ , \qquad (5)$$

where $C^i = \Gamma^0\Gamma^i$, and

$$D = \Gamma^0\left(-i\Gamma^4\partial_4 - i\Gamma^5\partial_5 + g\Phi\frac{(1-\Gamma_7)}{2} + g\Phi^*\frac{(1+\Gamma_7)}{2}\right) \ .$$

The operators C_i and D anticommute,

$$C_i D + DC_i = 0.$$

This means in particular that one can search for a solution of Eq. (5) as an expansion in the eigenvectors Ψ_m of the operator D,

$$D\Psi_m = m\Psi_m \ . \qquad (6)$$

There may exist a set of discrete eigenvalues m with separation of order gv, and continuous spectrum starting from $m \gtrsim gv$. For given k_μ, the eigenvalues m satisfy $k_0^2 = k_i^2 + m^2$. This means that to excite a mode with non-zero m, energy of order gv is required. In what follows we assume that the energy scales probed by a four-dimensional observer are much smaller than gv, and thus even the first non-zero level is not excited. So, we are interested only in zero modes of D:

$$D\Psi = 0 \; . \tag{7}$$

As is shown in Appendix B, the solution for $\Omega(x_\mu, x_4, x_5)$ has the form

$$\Omega(x) = \frac{1}{(2\pi)^{3/2}} \int d^4k \delta(k^2) \mathrm{e}^{-ik_\mu x^\mu} \sum_{p=0}^{k-1} a_{\mathbf{k}}^p \left(\begin{array}{c} 0 \\ f_p(r)\mathrm{e}^{ip\theta} c_{\mathbf{k}} \\ f_{k-p-1}(r)\mathrm{e}^{-i(k-p-1)\theta} c_{\mathbf{k}} \\ 0 \end{array} \right) \; , \tag{8}$$

where $c_{\mathbf{k}}$ is a normalized four-dimensional left-handed spinor, $a_{\mathbf{k}}^p$ are arbitrary complex functions of a three-dimensional momentum $\mathbf{k} = \{k_i\}$, and $f_l(r)$ is the normalized solution to the following differential equation,

$$f_l'' - \left(\frac{F'}{F} + \frac{k-1}{r} \right) f_l' + \left(\frac{F'}{F}\frac{l}{r} + \frac{l(k-l)}{r^2} - g^2 v^2 F^2 \right) f_l = 0. \tag{9}$$

At $r \to 0$, $f_l(r) \sim r^l$, while at $r \to \infty$, $f_l(r) \sim \mathrm{e}^{-gvr}$.

In other words, there exist k zero fermion modes with different angular and radial wave functions. One bulk fermion corresponds to k four-dimensional fermion species.

Let us emphasize that at given p, there is only one degree of freedom for particle ($k_0 > 0$) and one degree of freedom for antiparticle ($k_0 < 0$) that corresponds to one spin state of a left-handed spinor. Since $f_p(r)$ fall off exponentially at large r, the particles described by Ω are localized in the core of the vortex.

It is instructive to study how this eight-component localized spinor can be considered from the four-dimensional point of view. For this purpose, let us introduce a 8×8 representation of *four-dimensional* γ-matrices: $\tilde{\gamma}^\mu = \Gamma^\mu$,

$$\tilde{\gamma}^5 = i\Gamma_0\Gamma_1\Gamma_2\Gamma_3 = \left(\begin{array}{cc} -\gamma^5 & 0 \\ 0 & \gamma^5 \end{array} \right) \; .$$

These matrices together with the unit matrix, their commutators $\tilde{\sigma}^{\mu\nu} = \frac{i}{2}[\tilde{\gamma}^\mu \tilde{\gamma}^\nu]$, and $\tilde{\gamma}^\mu \tilde{\gamma}^5$ form an algebra $\tilde{\mathcal{A}}$ (that is any product of these matrices is their linear combination). The algebra $\tilde{\mathcal{A}}$ is isomorphic to algebra $\mathcal{A}$ of γ^μ, and one can construct an operator which relates $\tilde{\gamma}^\mu$ and γ^μ:

$$U^+ \tilde{\gamma} U = \gamma \ , \tag{10}$$

for any $\tilde{\gamma} \in \tilde{\mathcal{A}}$, $\gamma \in \mathcal{A}$. One can check that

$$U = \frac{1}{2} \begin{pmatrix} \gamma_0(1-\gamma_5) \\ 1+\gamma_5 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{11}$$

is one of the solutions to Eq.(10). To find four-component spinor $\omega(x)$ which corresponds to $\Omega(x)$, one should act by U^+ on Ω:

$$\begin{aligned} \omega(x) = U^+ \Omega(x) &= \frac{1}{(2\pi)^{3/2}} \int d^4k \delta(k^2) \mathrm{e}^{-ik_\mu x^\mu} \sum_{p=0}^{k-1} a^p_{\mathbf{k}} f_p(r) \mathrm{e}^{ip\theta} \begin{pmatrix} c_{\mathbf{k}} \\ 0 \end{pmatrix} \\ &\equiv \sum_{p=0}^{k-1} f_p(r) \mathrm{e}^{ip\theta} \psi_p(x_\mu) \ . \end{aligned}$$

We again obtain the same result: ω describes k left-handed massless spinors $\psi_p(x_\mu)$ localized at the vortex.

To localize k right-handed massless spinors at the vortex, one should consider six-dimensional spinor Ξ which has an axial charge $-1/2$ under $U(1)_g$: with obvious modifications, the above analysis goes through for Ξ as well.

4 Higgs field.

Let us introduce a complex scalar field H which interacts with the vortex field Φ:

$$\mathcal{L}_H = |\partial_A H|^2 - \frac{\kappa}{2} \left(|H|^2 - \mu^2 \right)^2 - h^2 |H|^2 |\Phi|^2. \tag{12}$$

Note that at this point H can have arbitrary charge under $U(1)_g$. Later on, we will set this charge to zero.

The system of the two scalar fields, Φ and H, described by the Lagrangian $\mathcal{L} = \mathcal{L}_\Phi + \mathcal{L}_H$, Eqs.(1) and (12), admits several nontrivial classical solutions. One of them corresponds to Φ given by Eq.(2) and $H = 0$. It can be shown [5], however, that this solution is unstable in a certain region of parameter space, in particular, for $\kappa\mu^2 \lesssim h^2 v^2$. We will consider this case in what follows; the lowest energy solution in the topological sector where Φ has the form (2) is

$$\begin{aligned} \Phi &= v\mathrm{e}^{ik\theta} F_c(r); \\ H &= H_c(r), \end{aligned} \tag{13}$$

where radial functions $F_c(r)$, $H_c(r)$ satisfy the following set of nonlinear differential equations,

$$\begin{aligned} F_c'' + \tfrac{1}{r} F_c' - \tfrac{k^2}{r^2} F_c - h^2 F_c H_c^2 - \lambda v^2 F_c \left(F_c^2 - 1\right) = 0, \\ H_c'' + \tfrac{1}{r} H_c' - h^2 H_c v^2 F^2 - \kappa H_c (H_c^2 - \mu^2) = 0. \end{aligned} \tag{14}$$

The boundary conditions are

$$F_c(0) = 0, \quad F_c(\infty) = 1;$$

$$H_c(0) = v_H \neq 0, \quad H_c(\infty) = 0,$$

where v_H is the vacuum expectation value of the Higgs field from four-dimensional point of view. The leading behavior of $H_c(r)$ near the origin is

$$H(r) = v_H \left(1 - \frac{\kappa(\mu^2 - v_H^2)}{4} r^2\right) . \tag{15}$$

So, to satisfy to the boundary condition at $r \to \infty$, it is required that $v_H < \mu$. The leading behavior of $F_c(r)$ is the same as of $F(r)$, Eq.(3). In the background of the modified vortex solution, Eq.(13), one still recovers all results of Sec. 3, replacing $F(r)$ by $F_c(r)$.

5 Four-dimensional fermion masses.

Let us turn to the fermion–Higgs Yukawa couplings. To describe a "prototype" generation, we need two six-dimensional spinors, Q and U, with opposite charges under $U(1)_g$:

$$Q \to \mathrm{e}^{i\frac{\alpha}{2}\Gamma_7} Q, \quad U \to \mathrm{e}^{-i\frac{\alpha}{2}\Gamma_7} U.$$

The interaction with the vortex with $k = 3$,

$$g_q \Phi \bar{Q} \frac{1-\Gamma_7}{2} Q + g_u \Phi^* \bar{U} \frac{1-\Gamma_7}{2} U + \text{h.c.}$$

results in the existence of three left– and right–handed zero modes of Q and U, respectively:

$$Q = \sum_{p_q=0}^{2} \frac{1}{(2\pi)^{3/2}} \int d^4k\, \delta(k^2) \mathrm{e}^{-ik_\mu x_\mu} a_{p_q}^{(q)}(\mathbf{k}) \begin{pmatrix} 0 \\ c_{\mathbf{k}} q_{p_q}(r) \mathrm{e}^{ip_q\theta} \\ c_{\mathbf{k}} q_{2-p_q}(r) \mathrm{e}^{-i(2-p_q)\theta} \\ 0 \end{pmatrix},$$

$$U = \sum_{p_u=0}^{2} \frac{1}{(2\pi)^{3/2}} \int d^4k\, \delta(k^2) \mathrm{e}^{-ik_\mu x_\mu} a_{p_u}^{(u)}(\mathbf{k}) \begin{pmatrix} d_{\mathbf{k}} u_{2-p_u}(r) \mathrm{e}^{-i(2-p_u)\theta} \\ 0 \\ 0 \\ d_{\mathbf{k}} u_{p_u}(r) \mathrm{e}^{ip_u\theta} \end{pmatrix}. \tag{16}$$

Here, $c_{\mathbf{k}}$ and $d_{\mathbf{k}}$ are the left-handed and right-handed four-dimensional spinors, respectively; arbitrary complex functions $a_{p_q}^{(q)}(\mathbf{k})$ for each $p_q = 0, 1, 2$ describe two degrees of freedom (one for the particle and one for the antiparticle) of a massless four-dimensional left-handed fermion Q_{p_q} with gauge and global quantum numbers of Q; $a_{p_u}^{(u)}(\mathbf{k})$ describe a massless four-dimensional right-handed fermion U_{p_u} with quantum numbers of U; $p_q = 0, 1, 2$ corresponds to three generations of fermions. The wave function profiles in extra dimensions are defined by radial functions $q_{p_q}(r)$, $u_{p_u}(r)$, which are the solutions of Eqs.(9) with $k = 3$, $f = q$ or u, $g = g_q$ or g_u, and $l = p_q$ or p_u, respectively.

Fermion masses originate from the following term in the six-dimensional action:

$$Y_u \int d^6x\, \tilde{H} \bar{Q} \frac{1-\Gamma_7}{2} U + \text{h.c.}$$

(Y_u is the six-dimensional Yukawa coupling constant, and $\tilde{H}_i = \epsilon_{ij} H_j^*$). This interaction is $U(1)_g$ invariant only if H is neutral under $U(1)_g$. To obtain the effective four-dimensional mass matrix, one has to perform the integration over extra dimensions in the action integral. This results in the Dirac mass terms,

$$m_{p_q p_u} \bar{Q}_{p_q} U_{p_u},$$

with

$$m_{p_q p_u} = Y_u \int r\, dr\, d\theta\, H_c(r) q_{p_q}(r) u_{p_u}(r) \mathrm{e}^{i(p_q - p_u)\theta}. \tag{17}$$

Integration over θ in Eq.(17) leads to a selection rule, $m_{p_q p_u} \sim \delta_{p_q p_u}$. This means that in this way it is possible to generate only diagonal mass terms, but not inter-generation mixings. To obtain mixings between fermions of different generations, it is necessary to have a non-trivial θ dependence in the Higgs mode. We will return to this point in Sec. 6.

To see that the hierarchy of diagonal mass values can be generated in this way, one can make use of the following, very rough, estimation. Characteristic distance scales for the Higgs and fermionic modes are $(\sqrt{\kappa}\mu)^{-1} \sim (hv)^{-1}$ and $(gv)^{-1}$, respectively. Let us take $h \gg g$, so that the Higgs mode is narrow in comparison to fermionic modes. Then one can substitute fermionic radial functions in Eq.(17) by their leading behavior at $r \to 0$,

$$q_p \sim (g_q v r)^p, \quad u_p \sim (g_u v r)^p$$

and to write $H_c = H_c(hvr)$. In this approximation, one gets

$$m \sim \left(\frac{g_q g_u}{h^2}\right)^p \equiv \delta_u^{2p}, \tag{18}$$

up to some slowly varying function on p,g, and h. Thus, at $\delta \sim 0.1$, mass hierarchy among three generations is naturally reproduced, if we associate the first generation with $p = 2$, the second one with $p = 1$ and the third one with $p = 0$. To obtain exact predictions, one has to solve the equations (9), (14) and to evaluate the radial integral in (17) numerically. A similar procedure yields masses of down–quarks and charged leptons, starting from interactions Eqs. (20), (21) (see below). Since dependence on g of wave function overlaps is highly nonlinear, the model does not predict any simple analytic relation between masses of fermions of different kinds (for example, $m_t/m_c \neq m_b/m_s$, etc.).

The parameters of the model with all Standard Model fermions included are: an overall mass scale, say, $Y_u\sqrt{\kappa}\mu$; two ratios of six-dimensional Yukawa couplings, Y_d/Y_u and Y_l/Y_u; and five ratios $g_{q,u,d,l,e}/h$. As follows from the discussion above (see Eq. (18)), to the leading approximation, masses depend only on three combinations of the latter five ratios, namely, $\delta_u = g_q g_u/h^2$, $\delta_d = g_q g_d/h^2$, and $\delta_l = g_l g_e/h^2$. Thus,

diagonal masses of nine charged fermions of the Standard Model arise from eight independent parameters, with significant dependence from six of them.

6 Mixing between generations.

To obtain off-diagonal mass matrix elements which mix fermions of different generations, one has to relax the selection rule $\delta_{p_q p_u}$ in Eq.(17) by introducing non-trivial θ dependence in the Higgs mode. To do this, one has to complicate the model, because the interaction (12) is phase–independent, so the classical solution H_c depends only on r. In what follows, we will need a somewhat more complicated topological defect, a global vortex made of two scalar fields with different winding numbers which appears in a model with

$$\mathcal{L} = \mathcal{L}_\Phi + \mathcal{L}_X,$$

where $\mathcal{L}_\Phi$ is presented in Eq.(1), and

$$\mathcal{L}_X = |\partial_A X|^2 - \frac{\lambda_1}{2}\left(|X|^2 - v_1^2\right)^2 - \alpha\left(X^3\Phi^* + X^{*3}\Phi\right).$$

Complex six-dimensional scalar fields Φ and X have charges 3 and 1 under global $U(1)_g$ symmetry, respectively*. Let $\tilde{v}$ and $\tilde{v}_1$ be vacuum expectation values of $|\Phi|$ and $|X|$, respectively;

$$\tilde{v} = v + O(\alpha),$$

$$\tilde{v}_1 = v_1 + O(\alpha).$$

At $\alpha < \lambda_1$, the model admits stable global vortices of the form

$$\begin{aligned} \Phi &= \tilde{v} \mathrm{e}^{3i\theta} \tilde{F}(r), \\ X &= \tilde{v}_1 \mathrm{e}^{i\theta} \chi(r), \end{aligned} \tag{19}$$

with $\tilde{F}(r)$ different than $F(r)$ in Eq.(2), but still having the same leading behavior at $r \to 0$ and $r \to \infty$. Note that in this model, the solution, Eq. (19), is the simplest nontrivial vortex.

Let us couple six-dimensional fermions corresponding to five fermionic fields of one generation of the Standard Model (namely, left-handed leptons L and quarks Q

*We assume that α is real; this can always be reached by rotating away its complex phase via redefinition of the origin of polar angle θ.

and right-handed leptons E, up quarks U and down quarks D) to Φ in a way similar to Eq.(4),

$$\begin{aligned} V_{\Phi f} &= g_q \Phi \bar{Q}_i \tfrac{1-\Gamma_7}{2} Q_i + g_l \Phi \bar{L}_i \tfrac{1-\Gamma_7}{2} L_i + \\ & g_u \Phi^* \bar{U} \tfrac{1-\Gamma_7}{2} U + g_d \Phi^* \bar{D} \tfrac{1-\Gamma_7}{2} D + g_e \Phi^* \bar{E} \tfrac{1-\Gamma_7}{2} E + \text{h.c.}, \end{aligned} \tag{20}$$

where we have written explicitly the electroweak $SU(2)$ indices i, and to the Higgs field H,

$$V_{Hf} = Y_u \tilde{H}_i \bar{Q}_i \frac{1-\Gamma_7}{2} U + Y_d H_i \bar{Q}_i \frac{1-\Gamma_7}{2} D + Y_l H_i \bar{L}_i \frac{1-\Gamma_7}{2} E + \text{h.c.}, \tag{21}$$

where $\tilde{H}_i = \epsilon_{ij} H_j$. Three chiral generations are localized in four dimensions as zero modes of the form (16) (with $F(r)$ replaced by $\tilde{F}(r)$ in corresponding equations).

On the other hand, let the Higgs field H couple to the field X in a way similar to Eq.(12),

$$V_{\Phi H} = \frac{\kappa}{2} \left(|H_i|^2 - \mu^2 \right)^2 + h^2 |H_i|^2 |X|^2 + \Delta V, \tag{22}$$

where ΔV is a small perturbation which breaks $U(1)_g$ global symmetry explicitly, for instance,

$$\Delta V = |H_i|^2 (\epsilon X + \epsilon^* X^*).$$

At $\epsilon = 0$, we recover Eq.(14) with $F_c(r)$ replaced by $\chi(r)$; its classical solution, call it $H_0(r)$, contributes to the diagonal masses through the integral Eq.(17). If $\epsilon \neq 0$, the solution to the nonlinear partial differential equation for $H(r,\theta)$ which follows from (22) in the background (19) has a complicated θ dependence. For our purposes, however, it is sufficient to perform perturbative expansion,

$$H(r,\theta) = H_0(r) + \epsilon H_1(r,\theta) + \epsilon^2 H_2(r,\theta) + \ldots + \text{h.c.} \tag{23}$$

The first term is $H_0(r)$ defined above. In the first order in ϵ, a source term of the form

$$\tilde{v}_1 \chi(r) \mathrm{e}^{i\theta} H_0(r),$$

appears in the equation for $H_1(r,\theta)$ which results in

$$H_1(r,\theta) = \mathrm{e}^{i\theta} h_1(r).$$

In a similar way,

$$H_2(r,\theta) = \mathrm{e}^{2i\theta} h_2(r).$$

The integral (17) reads now[†]

$$
\begin{aligned}
m^u_{ab} &= Y_u \int r\,dr\,d\theta\, H^*(r,\theta) q_{3-a}(r) u_{3-b}(r) \mathrm{e}^{i(b-a)\theta} = \\
&= Y_u (2\pi) \int r\,dr\, q_{3-a}(r) u_{3-b}(r) \quad \big(H_0(r)\delta_{ab} + \\
&\qquad \epsilon h_1(r)\delta_{a-1,b} + \epsilon^* h_1(r)\delta_{a,b-1} + \\
&\qquad \epsilon^2 h_2(r)\delta_{a-2,b} + \epsilon^{*2} h_2(r)\delta_{a,b-2} + \ldots\big)\,.
\end{aligned}
\tag{24}
$$

Non-diagonal mass matrix elements arise from overlaps of $h_{1,2}(r)$ with fermionic wave functions and are suppressed by powers of ϵ with respect to diagonal ones.

In the form discussed above, the model does not admit CP violation since the overall phase of Y_u is irrelevant and phase of ϵ can be included in two-component spinors $c_{\mathbf{k}}$ and $d_{\mathbf{k}}$ (see Eq.(16)). With more complicated ΔV, for example,

$$
\Delta V = \epsilon X |H_i|^2 + \epsilon_1 X^2 |H_i|^2 + \text{h.c.},
\tag{25}
$$

the relative phase of ϵ_1 and ϵ is a free CP violating parameter of the model (while all other results remain intact).

The relevant real parameters are now $g_{q,u,d,l,e}\tilde{v}/(h\tilde{v}_1)$, $|\epsilon|$, $|\epsilon_1|$, $\arg(\epsilon_1 - \epsilon)$ and again $Y_u\sqrt{\kappa}\mu$ and $Y_{d,l}/Y_u$. These eleven parameters of the six-dimensional theory thus generate thirteen parameters of the fermionic sector of the Standard Model (nine diagonal masses, three mixing angles, and one CP violating phase). As before, results depend significantly not on all five ratios $g\tilde{v}/(h\tilde{v}_1)$, but only on three combinations $g_q g_u \tilde{v}^2/(h^2\tilde{v}_1{}^2)$, $g_q g_d \tilde{v}^2/(h^2\tilde{v}_1{}^2)$, $g_l g_e \tilde{v}^2/(h^2\tilde{v}_1{}^2)$.

To demonstrate that it is possible to obtain a realistic hierarchical pattern of the fourteen Standard Model masses and mixings, let us estimate them in more detail. First of all, we find $H(r,\theta)$ (see Eq.(23)) in the model with ΔV given by Eq.(25). We assume that ϵ and ϵ_1 are of the same order and find $H(r,\theta)$ in the first order in ϵ, ϵ_1:

$$
H(r,\theta) = H_0(r) + h_1(r)(\epsilon \mathrm{e}^{i\theta} + \text{h.c.}) + h_2(r)(\epsilon_1 \mathrm{e}^{2i\theta} + \text{h.c.}).
\tag{26}
$$

Close to the origin

$$
H_0(r) = v_H; \quad h_1(r) = \frac{v_H X'(0)}{8} r^3; \quad h_2(r) = \frac{v_H X'(0)^2}{12} r^4,
\tag{27}
$$

[†]Hereafter, the indices $a, b = 1, 2, 3$ are used instead of $p_q, p_u, \ldots = 2, 1, 0$ to enumerate the fermionic generations; $a = 1$ corresponds to the first generation, etc.

(recall that X has winding number one and thus $X'(0) \neq 0$). On the other hand, the width of $h_{1,2}$ is of order of width of H_0 that is of order $(hv)^{-1}$. Substituting Eq.(27) into Eq.(24) and estimating all integrals as it has been done it in Sec. 5, we obtain the following mass matrix of up–quarks

$$m^u_{ab} \sim Y_u \begin{pmatrix} \delta_u^2 & \epsilon^* \delta_u^3 & \epsilon_1^* \delta_u^3 \\ \epsilon \delta_u^3 & \delta_u & \epsilon^* \delta_u^2 \\ \epsilon_1 \delta_u^3 & \epsilon \delta_u^2 & 1 \end{pmatrix}$$

and a similar matrix with the replacement $\delta_u \to \delta_d$, $Y_u \to Y_d$, for down quarks.

Cabibbo-Kobayashi-Maskawa (CKM) mixing matrix is defined by

$$U^{CKM} = S_u^\dagger S_d,$$

where S_u and S_d transform the matrices $m_u m_u^\dagger$ and $m_d m_d^\dagger$, respectively, to the diagonal form:

$$S^\dagger m m^\dagger S = \mathrm{diag}(\ldots).$$

With the definitions

$$A_u = \left| \frac{m^u_{21} m^u_{11} + m^u_{12} m^u_{22}}{m^{u2}_{22} - m^{u2}_{11}} \right| \sim |\epsilon| \delta^2 \ ; \quad B_u = \left| \frac{m^u_{23} m^u_{33} + m^u_{32} m^u_{22}}{m^{u2}_{33} - m^{u2}_{22}} \right| \sim |\epsilon| \delta^2 \ ;$$

$$C_u = \left| \frac{m^u_{13} m^u_{33} + m^u_{31} m^u_{11}}{m^{u2}_{33} - m^{u2}_{11}} \right| \sim |\epsilon_1| \delta^3 \ ;$$

and the same for down quarks we find that in the leading order,

$$\begin{aligned} U^{CKM} \simeq & \begin{pmatrix} 1 & (A_d - A_u) & (C_d - C_u) \\ (A_u - A_d) & 1 & (B_d - B_u)\mathrm{e}^{-i\varphi} \\ (C_u - C_d) & (B_u - B_d)\mathrm{e}^{i\varphi} & 1 \end{pmatrix} \\ \sim & \begin{pmatrix} 1 & |\epsilon|(\delta_d^2 - \delta_u^2) & |\epsilon_1|(\delta_d^3 - \delta_u^3) \\ |\epsilon|(\delta_u^2 - \delta_d^2) & 1 & |\epsilon|(\delta_d^2 - \delta_u^2)\mathrm{e}^{-i\varphi} \\ |\epsilon_1|(\delta_u^3 - \delta_d^3) & |\epsilon|(\delta_u^2 - \delta_d^2)\mathrm{e}^{i\varphi} & 1 \end{pmatrix} \end{aligned}$$

where $\varphi = \arg \epsilon_1 - 2 \arg \epsilon$. We see that there is a hierarchy in CKM matrix in our model which coincides with the hierarchy of CKM in Standard Model: the magnitude of the matrix elements decreases away from the diagonal, so that $V_{cb} \sim V_{us} \sim \epsilon \delta^2$, $V_{ub} \sim \epsilon_1 \delta^3$.

7 Conclusions.

The mechanism discussed here provides a consistent picture which can explain the origin of three generations of fermions with identical gauge and global quantum numbers but hierarchical mass matrices without fine tuning of parameters. Chiral fermionic zero modes are localized on a topological defect with topological number three which explains the origin of three generations; different masses appear due to different profiles of three fermionic zero modes in extra dimensions.

To explain mixings between generations, one has to introduce small but explicit violation of the global $U(1)_g$ symmetry of the theory. Off–diagonal mass matrix elements are suppressed by powers of the small parameter ϵ characterizing this violation. Though topological arguments are no longer valid in the presence of this $U(1)_g$ violation, we expect our results to be stable with respect to ϵ.

The model with a global vortex can be embedded either in a theory of large compact extra dimensions (in which case the problem of stability of the vortex in a compact space has to be addressed) or in a model with localized gravity [7]. The mechanism suggested here works as well in models with other topological defects with topological number three, for example, with a "hedgehog" in seven space-time dimensions.

The models which exploit the mechanism discussed above are in principle fairly predictive. For instance, in our particular toy model with a global vortex, eleven independent parameters of the six-dimensional theory generate thirteen masses and mixings of the Standard Model fermions, and the hierarchical structure of mass matrices is reproduced.

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A Notations.

Six-dimensional coordinates x_A are labeled by capital Latin indices $A, B = 0, \ldots, 5$. Four-dimensional coordinates x_μ are labeled by Greek indices $\mu, \nu = 0, \ldots, 3$; for spatial coordinates we use lower case Latin indices $i, j = 1, 2, 3$. The Minkowski metric is $g_{AB} = \mathrm{diag}(+, -, \ldots, -)$.

Dirac fermions in six dimensions are described by eight-component spinors; we work with the following representation of six-dimensional 8×8 Dirac matrices Γ^A:

$$\Gamma^A = \begin{pmatrix} 0 & \Sigma^A \\ \bar{\Sigma}^A & 0 \end{pmatrix} ,$$

where $\Sigma^0 = \bar{\Sigma}^0 = \gamma^0\gamma^0$; $\Sigma^i = -\bar{\Sigma}^i = \gamma^0\gamma^i$; $\Sigma^4 = -\bar{\Sigma}^4 = i\gamma^0\gamma^5$, $\Sigma^5 = -\bar{\Sigma}^5 = \gamma^0$, and γ^μ, γ^5 are the usual four-dimensional Dirac matrices in the chiral representation:

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} , \quad \gamma^5 = i\gamma_0\gamma_1\gamma_2\gamma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} ,$$

σ^i are the Pauli matrices.

We also introduce Γ_7 which is an analog of four-dimensional matrix γ_5

$$\Gamma_7 = \Gamma_0 \ldots \Gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

B Fermionic zero modes and their asymptotics.

The solution to Eq.(7) has the following structure

$$\Psi(k_\mu; r, \theta) = \begin{pmatrix} f_{(1)}(r) c_{(1)} \mathrm{e}^{i(p+1)\theta} \\ f_{(2)}(r) c_{(2)} \mathrm{e}^{ip\theta} \\ f_{(3)}(r) c_{(3)} \mathrm{e}^{-i(k-1-p)\theta} \\ f_{(4)}(r) c_{(4)} \mathrm{e}^{-i(k-p)\theta} \end{pmatrix} \tag{28}$$

In the last equation, p is an integer number; $c_{(a)}$ are two component columns (which, as will be shown below, correspond to four-dimensional chiral spinors), and $f_{(a)}$ satisfy the following set of differential equations ‡,

$$\begin{cases} f'_{(1)} + \dfrac{(p+1)}{r} f_{(1)} - gvF f_{(4)} = 0, \\ \\ f'_{(4)} + \dfrac{(k-p)}{r} f_{(4)} - gvF f_{(1)} = 0; \end{cases} \tag{29}$$

$$\begin{cases} f'_{(2)} - \dfrac{p}{r} f_{(2)} + gvF f_{(3)} = 0, \\ \\ f'_{(3)} - \dfrac{(k-p-1)}{r} f_{(3)} + gvF f_{(2)} = 0. \end{cases} \tag{30}$$

To investigate the behavior of $f_{(a)}$ and to find their asymptotics, it is convenient to introduce the new set of functions $\tilde{f}_{(a)}$:

$$f_{(1)} = r^{-(p+1)} \tilde{f}_{(1)} , \quad f_{(2)} = r^p \tilde{f}_{(2)} , \quad f_{(3)} = r^{(k-1-p)} \tilde{f}_{(3)} , \quad f_{(4)} = r^{-(k-p)} \tilde{f}_{(4)} . \tag{31}$$

The functions $\tilde{f}_{(a)}$ satisfy the following set of differential equations obtained from

‡These equations cannot be solved analytically, except for a particular case of $p = (k-1)/2$, when a normalized solution has the form

$$f_{(2),(3)} = \mathrm{const} \cdot r^{\frac{k-1}{2}} \exp\left(- \int^r gvF(x) dx \right) , \quad f_{(1),(4)} = 0 .$$

Eqs.(29), (30):

$$\left\{\begin{array}{l} \tilde f''_{(4)} - \left(\dfrac{F'}{F} + \dfrac{(k-2p-1)}{r}\right)\tilde f'_{(4)} - g^2v^2F^2\tilde f_{(4)} = 0, \\ \\ \tilde f_{(1)} = \dfrac{\tilde f'_{(4)}}{gvFr^{k-2p-1}}; \end{array}\right. \tag{32}$$

$$\left\{\begin{array}{l} \tilde f''_{(2)} - \left(\dfrac{F'}{F} + \dfrac{(k-2p-1)}{r}\right)\tilde f'_{(2)} - g^2v^2F^2\tilde f_{(2)} = 0, \\ \\ \tilde f_{(3)} = -\dfrac{\tilde f'_{(2)}}{gvFr^{k-2p-1}}. \end{array}\right. \tag{33}$$

Note that $\tilde f_{(1)}$ and $\tilde f_{(3)}$ satisfy the same differential equations as $\tilde f_{(4)}$ and $\tilde f_{(2)}$ respectively, with p replaced by $k-p-1$.

¿From Eqs.(32) it follows that, if at some point r_0 a solution $\tilde f_{(4)}$ and its derivative are positive ($\tilde f_{(4)}(r_0) \geq 0$, $\tilde f'_{(4)}(r_0) > 0$), then $\tilde f_{(4)}$ increases with r at any point $r > r_0$. To see this, let us first note that equation on $f_{(a)}$ is nothing but a Schrödinger type equation for the lowest energy level. This means that $f_{(a)}$ cannot have nodes at $0 < r < \infty$, so we can assume $f_{(4)} \geq 0$. Let us assume also that $(k-1)/2 \geq p$, and thus $k > p$. Since $F'/F \geq 0$ and $\tilde f'_{(4)}(r_0) > 0$, it follows from Eq.(32) that $\tilde f''_{(4)}(r_0) > 0$. It means that $\tilde f'_{(4)}$ increases and so $\tilde f'_{(4)}(r) > 0$ at any point $r > r_0$.

Let us now study the behavior of $\tilde f_{(a)}$ at $r \to 0$. In this limit, $F \simeq r^k$ and from Eqs. (32) and (33) it follows that

$$\tilde f_{(4)} \simeq r^0 \;\text{ or }\; \tilde f_{(4)} \simeq r^{2(k-p)} \;;\;\; \tilde f_{(1)} \simeq r^0 \;,$$

$$\tilde f_{(2)} \simeq r^0 \;\text{ or }\; \tilde f_{(2)} \simeq r^{2(k-p)} \;;\;\; \tilde f_{(3)} \simeq r^0 \;.$$

Since $k > p$, the solutions which have the behavior $\tilde f_{(2),(4)} \simeq r^{2(k-p)}$ increase everywhere.

Now let us consider asymptotics of $\tilde f_{(a)}$ at $r \to \infty$, when $F \sim 1$ and first equation in (32) reads

$$\tilde f''_{(4)} - \frac{k-2p-1}{r}\tilde f'_{(4)} - g^2v^2\tilde f_{(4)} = 0 \;.$$

This equation has two linearly independent solutions,

$$\tilde f^{I}_{(4)} = r^{(k-2p)/2}K_{(k-2p)/2}(gvr) \;;\;\; \tilde f^{II}_{(4)} = r^{(k-2p)/2}I_{(k-2p)/2}(gvr) \;,$$

where $I_\nu(r)$ and $K_\nu(r)$ are modified Bessel functions of the first and the third order. The second solution $\tilde{f}^{II}_{(4)}$ grows exponentially at infinity. On the other hand, as we have shown above, if a solution to equations (32), (33) increases at some r, then it increases everywhere. This means that the solution which behaves as $\tilde{f}_{(4)} \simeq r^{2(k-p)}$ at $r \to 0$ has the asymptotic behavior $\tilde{f}^{II}_{(4)}$ at $r \to \infty$. Thus, the corresponding function is not normalizable. Therefore, all functions $\tilde{f}_{(a)}$ which can correspond to normalizable solutions have the same asymptotics: they tend to constant at $r = 0$ and exponentially fall off at infinity. This results in the following behavior of functions $f_{(a)}$: they exponentially fall off at infinity,

$$f_{(a)} \propto \mathrm{e}^{-gvr} \ , \quad r \to \infty \ , \tag{34}$$

and close to the origin,

$$f_{(1)} \simeq r^{-(p+1)} \ , \quad f_{(4)} \simeq r^{-(k-p)} \ , \quad f_{(2)} \simeq r^{p} \ , \quad f_{(3)} \simeq r^{k-p-1} \ , \quad r \to 0 \ . \tag{35}$$

Since $k-p > 0$, $f_{(4)}$ is not normalizable: the corresponding integral diverges at $r = 0$. So, one is forced to conclude that $f_{(1)} = f_{(4)} = 0$. Moreover, to allow $f_{(2)}$ to be normalizable one should require that $p \geq 0$.

The case $2p \geq k-1$ can be studied in the same way: instead of functions $\tilde{f}_{(4)}$ and $\tilde{f}_{(2)}$, one should consider functions $\tilde{f}_{(1)}$ and $\tilde{f}_{(3)}$. As a result one finds that $k-1 \geq p$. There are k linearly independent normalizable solutions which can be labeled by index $p = 0, \ldots, k-1$:

$$\Psi_p = \begin{pmatrix} 0 \\ f_p(r)\mathrm{e}^{ip\theta}c_p \\ f_{k-1-p}(r)\mathrm{e}^{-i(k-p-1)\theta}c_{k-p-1} \\ 0 \end{pmatrix} ,$$

where f_p are the normalized solutions of the equations (30) which behave as r^p at zero and fall exponentially at infinity.

Substituting spinor Ψ_p into equation

$$k_0\Psi_p = C_i k_i \Psi_p \ ,$$

one finds that c_p and c_{k-p} satisfy the following equation

$$(k_0 + k_i\sigma_i)c_p = (k_0 + k_i\sigma_i)c_{k-p} = 0 \ ,$$

which defines a left-handed four-dimensional spinor for $k_0 > 0$. This equation has one solution (iff $k_0^2 = k_i^2$) which we will denote as $c_{\mathbf{k}}$. To conclude, we have precisely k solutions which describe k massless left-handed four-dimensional fermions in full agreement with the index theorem, so the solution to Eq.(5), which corresponds to the zero mode, has the form (8).

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